Modern approaches to quantum gravity

Homework 10 Fall 2025

1. BTZ black hole

In three-dimensional anti-de Sitter spacetimes, there exist black hole solutions, known as BTZ (Bañados, Teitelboim and Zanelli) black holes. The metric of the non-rotating BTZ black hole is given by

$$ds^2 = -\left(\frac{r^2}{\ell^2} - 8GM\right) dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} - 8GM} + r^2 d\phi^2. \tag{1}$$

(a) Show that this spacetime is locally AdS_3.

(b) Find the global Killing vectors of this spacetime. How many are they?

(c) Show that the hypersurface $r = r_h = \ell\sqrt{8GM}$ is a Killing horizon.

(d) Compute the energy of this spacetime, using your preferred formalism.

(e) Show that the surface gravity at the horizon is $\kappa = r_h/\ell^2$.

(f) Show the same can be obtained by the Euclidean continuation of the metric, and demanding smoothness at r_h.

(g) Compute the Bekenstein-Hawking entropy of this black hole.

(h) Using modular invariance, it can be shown that the entropy of a 2d CFT with central charge c at large E is given by

$$S_{\text{Cardy}} = 2\pi\sqrt{\frac{c\ell E}{3}}. \tag{2}$$

This is known as the Cardy entropy and it is a universal formula for the density of states in a 2d CFT at high energies, which is valid irrespective of whether the theory is weakly or strongly coupled. Show this formula precisely reproduces the entropy of the BTZ black hole.

(i) *(Optional)* Consider now the rotating BTZ black hole with mass M and angular momentum J.

$$ds^2 = -N(r)^2\, dt^2 + \frac{dr^2}{N(r)^2} + r^2\big(d\varphi + N^\varphi(r)\, dt\big)^2, \tag{3}$$

with

$$N(r)^2 = -8GM + \frac{r^2}{\ell^2} + \frac{4G^2J^2}{r^2}, \qquad N^\varphi(r) = -\frac{2GJ}{r^2}. \tag{4}$$

Find the inner and outer horizons, the horizon angular velocity Ω_H, and the temperature of the black hole. Compute the entropy S, and verify that the entropy agrees with the Cardy formula S_{Cardy} given by

$$S_{\text{Cardy}} = 2\pi\sqrt{\frac{c}{6}\ell E_L} + 2\pi\sqrt{\frac{c}{6}\ell E_R} \tag{5}$$

where $M = E_L + E_R$ and $J = \ell(E_L - E_R)$.

2. The Cardy formula

The partition function of a 2d QFT on a (rectangular) torus of periodicities β and $2\pi R$ can be given a Hilbert space interpretation as either $\operatorname{Tr} e^{-\beta H}$ for the QFT on a circle of radius R, or as $\operatorname{Tr} e^{-2\pi R H}$ for the QFT on a circle of radius $\beta/2\pi$:

$$Z(\beta, R) = Z\left(2\pi R, \frac{\beta}{2\pi}\right). \tag{18}$$

(a) Show that when applied to a 2d CFT, this implies a relation between the high-temperature and low-temperature partition function

$$Z_{\text{CFT}}(\beta, R) = Z_{\text{CFT}}\left(\frac{4\pi^2 R^2}{\beta}, R\right). \tag{19}$$

(b) Use this relation to show that the high energy density of states in a CFT_2 is compatible with the Cardy formula (2). (Hint: what is the lowest-energy state in a unitary 2d CFT?)

(c) *(Optional)* For simplicity let us now set the circle radius to 1. This idea of swapping the two circles of the CFT_2 partition function generalizes to the so-called modular transformations. Define the quantity

$$Z(\tau, \bar{\tau}) = \text{Tr} e^{-2\pi\tau_2 H + 2\pi i \tau_1 J} \tag{6}$$

where $H = L_0 + \bar{L}_0 - \frac{c}{12}$ is the Hamiltonian on the cylinder, and $J = L_0 - \bar{L}_0$ generates translations along the circle. We also defined $\tau \equiv \tau_1 + i\tau_2$ and $\bar{\tau} \equiv \tau_1 - i\tau_2$. This object Z can be interpreted as a Euclidean 2d CFT path integral over the torus in the following manner.[1] Remind that a torus can be viewed as a parallelogram with its parallel sides periodically identified. Writing it in complex coordinates $z = \sigma^1 + i\sigma^2$, and using conformal transformations, we can write the torus identification as $(\sigma^1, \sigma^2) \sim (\sigma^1 + 2\pi, \sigma^2)$ and $(\sigma^1, \sigma^2) \sim (\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2)$. Note that τ_1 and τ_2 are free parameters that cannot be continuously modified by conformal transformations. We call the combination τ the modulus of the torus. With this in mind, one should view the torus as a lattice in the complex plane, formed by the lattice points $n_1 + n_2\tau$ with $n_1, n_2 \in \mathbb{Z}$, where each cell is related to one another by the identification $z \sim z + 1$ and $z \sim z + \tau$. From that point of view, it is clear that for example τ and $\tau + 1$ represent the same torus. Convince

[1] For more details, you may have a look at Polchinski's Volume 1 String Theory book. The torus is described in section 5.1, and the torus partition function is discussed around equation (7.2.5).

yourself that τ and $-1/\tau$ also represent the same torus. The combination of these two discrete transformations ($\tau \to -1/\tau$ and $\tau \to \tau + 1$) generates the so-called modular transformations (the group is $SL(2, \mathbb{Z})$).

The object $Z(\tau, \bar{\tau})$ we defined above precisely agrees with the CFT$_2$ partition function on a torus of modulus τ. To see it, remember that $\text{Tr}e^{-2\pi\tau_2 H}$ leads to the usual time identification $\sigma^2 \sim \sigma^2 + 2\pi\tau_2$ in the path integral, but before gluing the final time slice to the initial time slice, the translation operator $e^{2\pi i\tau_1 J}$ rotates $\sigma^1 \to \sigma^1 + 2\pi\tau_1$, so the gluing is actually shifted, yielding the identification $(\sigma^1, \sigma^2) \sim (\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2)$. This precisely corresponds to the path integral over a torus of modulus τ. The modular invariance of the torus discussed above then implies[2]

$$Z(\tau, \bar{\tau}) = Z(-1/\tau, -1/\bar{\tau}) \tag{7}$$

Using this fact and the knowledge of the Casimir energy, show that the thermal partition function in the ensemble at temperature $T = 1/\beta$ and angular velocity Ω

$$Z(\beta, \Omega) = \text{Tr}e^{-\beta(H - \Omega J)} \tag{8}$$

can be approximated at high temperature as

$$\log Z \approx \frac{\pi i c}{12}\left(\frac{1}{\tau} - \frac{1}{\bar{\tau}}\right) \tag{9}$$

where $2\pi i\tau \equiv \beta\Omega - \beta$ and $2\pi i\bar{\tau} = \beta\Omega + \beta$. Show that the entropy in the microcanonical ensemble at fixed and very large $E_L = \langle L_0 - \frac{c}{24}\rangle$ and $E_R = \langle \bar{L}_0 - \frac{c}{24}\rangle$ takes the form

$$S_{\text{Cardy}} = 2\pi\sqrt{\frac{c}{6}E_L} + 2\pi\sqrt{\frac{c}{6}E_R} \tag{10}$$

Note that E_L and E_R are related to more standard quantities (energy E and angular momentum J) by $\langle E\rangle = E_L + E_R$ and $\langle J\rangle = E_L - E_R$.

3. Asymptotic charges in gauge theory

Consider the action for a free complex scalar ϕ, and the corresponding action for the complex scalar coupled to the Maxwell field A_μ, living on a d-dimensional manifold M

$$S_1 = \int_M d^dx\ \partial_\mu\phi^\star\partial^\mu\phi \tag{11}$$

$$S_2 = \int_M d^dx\ D_\mu\phi^\star D^\mu\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{12}$$

$$\tag{13}$$

where $D_\mu \equiv \partial_\mu - igA_\mu$.

[2] Note that strictly speaking, our previous argument shows that modular invariance holds for $\bar{z} = z^*$. However, assuming analyticity, this can be extended to all values of $z, \bar{z}$.

(a) Compute the current associated to the global $U(1)$ symmetry of S_1, and the corresponding conserved charge Q_1 on a spacelike hypersurface Σ.

(b) Let us start by reviewing the concepts defined in the lecture. Remember that a generic variation of the action takes the form

$$\delta S = \int d^d x (E\delta\phi + \partial_\mu \Theta^\mu) \tag{14}$$

where E denotes the equations of motion. If the variation δ_ϵ is a (global or local) symmetry, that means that for generic field configurations,

$$\delta_\epsilon S = \int d^d x \partial_\mu M^\mu \tag{15}$$

Now restrict yourself to local symmetries whose form obey

$$\delta_\epsilon \phi = f(\phi)\epsilon(x) + f^\mu(\phi)\partial_\mu \epsilon(x) \tag{16}$$

By considering ϵ to have local support, and using the above, argue that local invariance implies the Noether identity

$$Ef(\phi) - \partial_\mu (E f^\mu(\phi)) = 0 \tag{17}$$

Argue that

$$E\delta_\epsilon \phi = -\partial_\mu(\Theta^\mu - M^\mu) \equiv -\partial_\mu J^\mu \tag{18}$$

and using the Noether identity prove that

$$\partial_\mu J^\mu = -\partial_\mu \underbrace{(E f^\mu \epsilon)}_{\equiv S^\mu} \tag{19}$$

Conclude that

$$J^\mu = -S^\mu - \partial_\nu Q^{\mu\nu} \tag{20}$$

where $Q^{\mu\nu}$ must be antisymmetric. Note that S^μ vanishes on-shell.

(c) Now back to our concrete example of scalar QED with action S_2, consider a local variation

$$\phi \to e^{ig\epsilon(x)}\phi, \quad A_\mu \to A_\mu + \partial_\mu \epsilon(x) \tag{21}$$

with local parameter $\epsilon(x)$. Compute the Noether current J^μ and show that it is a pure derivative, as predicted above.

(d) *(Optional)* Write down the Noether identity for scalar QED.

(e) Show that the charge Q_2 associated to this current on the manifold Σ can now be expressed as a boundary integral on $\Sigma' \equiv \partial\Sigma$. Interpret this result, and compare it to the one from the first point.

(f) Show that if $\epsilon(x)$ is taken to vanish on Σ', then $Q_2 = 0$. In light of this, determine what are the true symmetries of S_2, and in particular their relation to their effect on the boundary.

(g) What happens when we take the gauge coupling $g \to 0$?